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Representation Theorem of Observables on a Quantum System

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An orthomodular lattice (OML) with a conditional state can be used as a model for noncompatible events (a quantum system). In this paper we will study some properties of a conditional state and an *s*-map which are defined on an OML. We show conditions when a quantum system has the same properties as the classical probability space.

KEY WORDS: quantum logic; orthomodular lattice; statte; conditional state.

1. INTRODUCTION

The idea that quantum probabilities are nothing else than conditional probabilities was intensively discussed from various points of view, see, e.g., (Accardi, 1984; Ballentine, 1986, 2001; Beltrametti and Cassinelli, 1981; Cox, 1961; De Muynck, 2001; Dvurečenskij and Pulmannová, 2000; Gudder, 2001; Nánásiová, 1987; Pták and Pulmannová, 1991; Varadarajan, 1968). Recently this approach to quantum probability was generalized in the so called contextual framework (Khrennikov, 1999, 2000, 2001a,b, 2004; Nánásiová, 1987a,b, 1993a,b). In the latter approach we consider conditioning with respect to various contexts, complexes of experimental physical conditions and not conditioning of one event a, with respect to other event b, under the same context (state) as it was done in investigations on conditional probabilities (compare with Kolmogorov (1993, 1965), Renyi (2006), Cox (1961), or quantum system generalizations (Baltrametti, 2001; Baltrametti and Cassinelli, 1981; Nánásiová, 1987a,b, 1993a)).

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In the present paper we still use the traditional event-conditioning (under the fixed conditional state). However, we essentially generalize the notion of a conditional state (here we continue investigations (Nánásiová, 1993b, 1998, 2000, 2006a,b)).

We will study a conditional state on a quantum system using Renyi's approach. This approach helps us to define an independence of events in a different way, than it is done in the classical theory of probability. If an event a is independent of an event b, then the event b can be dependent on the event a (problem of causality) (Nánásiová, 1998, 2006a,b)). We can define an s-map (function for simultaneous measurements on a quantum system). By using the s-map we can introduce a joint distribution also for noncompatible observables on a quantum system. It can be shown that we can define a covariance and a correlation on L. We show, that also for an un-symmetry s-map on a Boolean algebra (a single probability space) we get the same situation as in the classical theory of probability.

1.1. A Conditional State on an OML

In this part we introduce the notions as a quantum system (an orthomodular lattice), a state, a conditional state, and their basic properties.

Definition 1.1. Let L be a nonempty set endowed with a partial ordering $\leq$. Let there exist the greatest element 1 and the smallest element 0. We consider operations supremum ($\vee$), infimum $\wedge$ (the lattice operations), and an map $\perp : L \to L$ defined as follows.

(i) For any $\{a_n\}_{n\in A} \in L$, where $\mathcal{A} \subset \mathcal{N}$ is finite,

$$\bigvee_{n\in\mathcal{A}} a_n, \bigwedge_{n\in\mathcal{A}} a_n \in L.$$

(ii) For any $a \in L (a^\perp)^\perp = a$.
(iii) If $a \in L$, then $a \vee a^\perp = 1$.
(iv) If $a, b \in L$ such that $a \leq b$, then $b^\perp \leq a^\perp$.
(v) If $a, b \in L$ such that a $\leq$ b then $b = a \vee (a^\perp \wedge b)$ (orthomodular law).

Then $(L, 0, 1, \vee, \wedge, \perp)$ is said to be the orthomodular lattice (briefly OML). Let L be an OML. Then elements $a, b \in L$ will be called:

- orthogonal ($a \perp b$) iff $a \leq b^\perp$;
- compatible ($a \leftrightarrow b$) iff there exist mutually orthogonal elements $a_1, b_1, c \in L$ such that

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c.$$

If $a_i \in L$ for any $i = 1, 2, \ldots, n$ and $b \in L$ is such, that $b \leftrightarrow a_i$ for all i, then $b \leftrightarrow \bigvee_{i=1}^{n} a_i$ and

$$b \wedge \left(\bigvee_{i=1}^{n} a_i \right) = \bigvee_{i=1}^{n} (a_i \wedge b)$$

(Dvurečenskij and Pulmannová, 2000; Pták and Pulmannová, 1991; Varadarajan, 1968).

Definition 1.2. A map $m : L \to [0, 1]$ such that

(i) $m(0) = 0$ and $m(1) = 1$.
(ii) If $a \perp b$ then $m(a \vee b) = m(a) + m(b)$ is called a state on L.

Definition 1.3. Nánásiová (2006a). Let L be an OML. A subset $L_0 \subset L - \{0\}$ is called a conditional system (CS) in L if the following conditions hold:

- If $a, b \in L_0$, then $a \vee b \in L_0$.
- If $a, b \in L_0$ and $a < b$, then $a^{\perp} \wedge b \in L_0$.

Let $A \subset L$. Then $L_0(A)$ is the smallest CS, that contains the set A.

Definition 1.4. (Nánásiová, 2006a). Let L be an OML and let L_0 be a CS in L. Let $f : L \times L_0 \to [0, 1]$. If the function f fulfills the following conditions:

(C1) for each $a \in L_0 f(., a)$ is a state on L;
(C2) for each $a \in L_0 f(a, a) = 1$;
(C3) if $\{a_i\}_{i=1}^{n} \in L_0$ and a_i are mutually orthogonal, then for each $b \in L$

$$f\left(b, \bigvee_{i=1}^{n} a_i\right) \sum_{i=1}^{n} f\left(a_i, \bigvee_{i=1}^{n} a_i\right) f(b, a_i);$$

then it is called a conditional state.

Proposition 1.1. *(Nánásiová, 2006a). Let L be an OML. Let $\{a_i\}_{i=1}^{n} \in L$, $n \in N$ where $a_i \perp a_j$ for $i \neq j$. If for any i there exists a state α_i, such that $\alpha_i(a_1) = 1$, then there exists a CS such that for any $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, where $k_i \in [0; 1]$ for $i \in \{1, 2, \ldots, n\}$ with the property $\sum_{i=1}^{k} k_i = 1$, there exists a conditional state*

$$f_{\mathbf{k}} : L \times L_0 \to [0; 1],$$

such that

1. for any i and each $d \in L f_{\mathrm{k}}(d, a_i) = \alpha_i(d)$;

2. for each a_i

$$f_k\left(a_i, \bigvee_{i=1}^{n} a_i\right) = k_i.$$

Proposition 1.2. *(Nánásiová, 2006a) Let L be an OML and f be a conditional state. Let $b \in L$, $a, c \in L_0$ such that $f(c, a) = 1$. Then b is independent of a with respect to the state $f(., c)(\asymp_{f(.,c)} a)$ if $f(b, c) = f(b, a)$.*

The classical definition of independency of events in a probability space $(\Omega, \mathcal{B}, \mathcal{P})$ is a special case of this definition, because

$$P(A|B) = P(A|\Omega) \text{ and only if } P(A \cap B|\Omega) = P(A|\Omega)P(B|\Omega).$$

If L_0 be CS and $f : L \times L_0 \to [0, 1]$ is a conditional state, then (Nánásiová, 2006a)

(i) Let $a^\perp, a, c \in L_0, b \in L$ and $f(c, a) = f(c, a^\perp) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $b \asymp_{f(.,c)} a^\perp$.
(ii) Let $a, c \in L_0, b \in L$ and $f(c, a) = 1$. Then $b \asymp_{f(.,c)}$ a if and only if $b^\perp \asymp_{f(.,c)} a$.
(iii) Let $a, c, b \in L_0, b \leftrightarrow a$ and $f(c, a) = f(c, b) = 1$. Then $b \asymp_{f(.,c)} a$ if and only if $a \asymp_{f(.,c)} b$.

2. OBSERVABLES AND AN *S*-MAP

Let L be an OML. Let us denote $L^2 = L \times L$.

Definition 2.1. Nánásiová (2006b). Let L be an OML. The map $p : L^2 \to [0, 1]$ will be called an s-map if the following conditions hold:

(s1) $p(1, 1) = 1$;
(s2) if $a \perp b$, then $p(a, b) = 0$;
(s3) if $a \perp b$, then for any $c \in L$,

$$p(a \vee b, c) = p(a, c) + p(b, c)$$

$$p(c, a \vee b) = p(c, a) + p(c, b).$$

Proposition 2.1. *Nánásiová (2006b). Let L be an OML and let there be an s-map p. Let $a, b, c \in L$, then*

1. *if $a \leftrightarrow b$, then $p(a, b) = p(a \wedge b, a \wedge b) = p(b, a)$;*
2. *if $a \leq b$, then $p(a, b) = p(a, a)$;*

3. *if* $a \leq b$, *then* $p(a, c) \leq p(b, c)$*;*
4. $p(a, b) \leq p(b, b)$*;*
5. *if* $\nu(b) = p(b, b)$*, then* ν *is a state on* L*.*

Proposition 2.2. *Nánásiová (2006b). Let* L *be an OML, let there be an* s*-map* p*. Then there exists a conditional state* f_p*, such that*

$$p(a, b) = f_p(a, b) f_p(a, 1).$$

Let L *be an OML and let* $L_0 = L - \{0\}$*. If* $f : L \times L_0 \to [0, 1]$ *is a conditional state, then there exists an* s*-map* $p_f : L \times L \to [0, 1]$*.*

Indeed in the Nánásiová (2006b) has been shown that if p is a s-map and $L_0 = \{b \in L : p(b, b) \neq 0\}$, then $f_p(a, b) = \frac{p(a,b)}{p(b,b)}$ is a conditional state and conversely if $L_0 = L - \{0\}$, then $p_f(a, b) = f(a, b) f(b, 1)$ is a s-map.

Proposition 2.3. *Nánásiová (2006b). Let* L *be an OML.*

(a) If f *is a conditional state, then* $b \asymp_{f(.,1)} a$ *iff* $p_f(b, a) = p_f(a, a) p_f(b, b)$*, where* p_f *is the* s*-map generated by* f*.*
(b) Let p *be an* s*-map. Then* $b \asymp_{f_p(.,1)} a$ *iff* $p(b, a) = p(a, a) p(b, b)$*, where* f_p *is the conditional state generated by the* s*-map* p*.*

We say that a s-map p is an un-symmetric s-map if there exist $a, b \in L$, such that $p(a, b) \neq p(b, a)$. If for each $a, b \in L p(a, b) = p(b, a)$ we say that p is a symmetric s-map.

Proposition 2.4. *Let* L *be an OML and let* p *be an un-symmetric* s*-map. Then there exist two symmetric* s*-maps* q_1 *and* q_2 *such that for any* $c \in L q_i(c, c) = \nu(c), i = 1, 2$*.*

Proof: Let p be an un-symmetric s-map. If for $c, d \in L p(c, d) = p(d, c)$, then $q_i(c, d) = p(c, d), i = 1, 2$. From the assumption it follows that there exist $a, b \in L$, such that $p(a, b) \neq p(b, a)$. Let $q_1(a, b) = q_1(b, a) = p(a, b)$ and $q_2(a, b) = q_2(b, a) = p(b, a)$. From it follows, that q_i is symmetric, $i = 1, 2$. As p is the symmetric s-map for each compatible elements (Proposition 2.1), then q_i is also s-map and $q_i(c) = \nu(c)$ for $c \in L$ and $i = 1, 2$. □

We say that a conditional state $f(., .)$ is Bayesian if for each $a, b \in L_0 f(a, b) f(b, 1) = f(b, a) f(a, 1)$.

Proposition 2.5. *Let* L *be an OML. Let* p *be a* s*-map and let* f *be a conditional state.*

(a) *The* s*-map* p_f *is symmetric for each* $a, b \in L_0$ *iff* f *is the Bayesian conditional state.*
(b) *The* s*-map* p *is symmetric iff* f_p *is the Bayesian conditional state.*

Proof: (a) Let f be a Bayesian conditional state and $p_f(a, b) = f(a, b)f(b, 1)$. From it follows, that

$$p_f(a, b) = f(a, b)f(b, 1) = f(a, b)f(a, 1) = p_f(b, a).$$

Conversely, if $p_f(a, b) = p_f(b, a)$, then

$$f(a, b)f(b, 1) = p_f(a, b) = p_f(b, a) = f(b, a)f(a, 1).$$

(b) Let $p(a, b) = p(b, a)$ for each $a, b \in L$. Let $a, b \neq 0$. Then $f_p(a, b) = p(a, b)/p(b, b)$ and $f_p(b, 1) = p(b, b)$. From it follows, that

$$p(b, b)f_p(a, b) = f_p(b, 1)f_p(a, b) = p(a, b)$$

$$p(a, a)f_p(b, a) = f_p(a, 1)f_p(b, a) = p(b, a).$$

Hence p is the symmetric s-map, then

$$f_p(b, 1)f_p(a, b) = f_p(a, 1)f_p(b, a).$$

The opposite implication we can prove analogically. □

Let $\mathcal{B}(\mathcal{R})$ be a σ-algebra of Borel sets. A homomorphism $x : \mathcal{B}(\mathcal{R}) \to L$ is called an observable on L. If x is an observable, then $R(x) := \{x(E); E \in \mathcal{B}(\mathcal{R})\}$ is called a range of the observable x. It is clear that $R(x)$ is a Boolean algebra [Var]. A spectrum of an observable x is defined by the following way: $\sigma(x) = \cap\{E \in \mathcal{B}(\mathcal{R}); x(E) = 1\}$. If g is a real function, then $g \circ x$ is such observable on L that:

1. $R(g \circ x) \subset R(x)$;
2. $\sigma(g \circ x) = \{g(t); t \in \sigma(x)\}$;
3. for any $E \in \mathcal{B}(\mathcal{R})$
   $g \circ x(E) = x(\{t \in \sigma(x); g(t) \in E\})$.

We say that x and y are compatible ($x \leftrightarrow y$) if there exists a Boolean sub-algebra $\mathcal{B} \subset L$ such that $R(x) \cup R(y) \subset \mathcal{B}$. In other words $x \leftrightarrow y$ if for any $E, F \in, \mathcal{B}(\mathcal{R}), x(E) \leftrightarrow y(F)$.

We call an observable x a finite if $\sigma(x)$ is a finite set. It means, that $\sigma(x) = \{t_i\}_{i=1}^n, n \in N$. Let us denote $\mathcal{O}$ the set of all finite observables on L.

Definition 2.2. Let L be an OML and $p : L \times L \to [0; 1]$ be an s-map. Let $x, y \in \mathcal{O}$. Then an map $P_{x,y} : \mathcal{B}(\mathcal{R}) \times \mathcal{B}(\mathcal{R}) \to [0, 1]$, such that

$$P_{x,y}(E, F) = p(x(E), y(F)),$$

is called a joint distribution for the observables x and y for the s-map p.

If $x \in \mathcal{O}$ and m is a state on L, then $m_x(E) = m(x(E))$, $E \in \mathcal{B}(\mathcal{R})$ is a probability distribution for x and

$$m(x) = \sum_{t \in \sigma(x)} t m(x(t))$$

and for any real function g we have

$$m(g \circ x) := \sum_{t \in \sigma(x)} g(t) m(x(t)).$$

Definition 2.3. Let L be an OML and let $p : L \times L \to [0; 1]$ be an s-map. Let $x, y, \in \mathcal{O}$ Then

$$p(x, y) = \sum_{x_i \in \sigma(x)} \sum_{y_j \in \sigma(y)} x_i y_j p(x(x_i), y(y_j)),$$

is called the first joint moment for observables x, y.

From the previous definition it follows that

$$p(x, x) = \sum_{x_i \in \sigma(x)} x_i^2 p(x(x_i), x(x_i)) = \sum_{x_i \in \sigma(x)} x_i^2 \nu(x(x_i)) := \nu(g \circ x),$$

where $g(t) = t^2$.

From analogy with the classical theory of probability we can define notions for example as covariance ($c(.,.)$), variance (var(.)) and correlation coefficient ($r(.,.)$) by the following way:

$$c(x, y) := p(x, y) - \nu(x)\nu(y),$$
$$\mathrm{var}(x) := c(x, x),$$
$$r(x, y) = \frac{c(x, x)}{\sqrt{\mathrm{var}(x)\mathrm{var}(y)}}.$$

In spite of the classical theory of probability in this case $c(x, y)$ is not equal to $c(y, x)$ in general.

Proposition 2.6. *Let L be an OML, let p be an s-map on L and let $\mathcal{O}$ be a set of all finite observables on L. For each $x, y \in \mathcal{O}$ there exist probability spaces $(\Omega, \mathcal{S}_i, P_i)(i = 1, 2)$ and random variables $\xi_i, \eta_i (i = 1, 2)$, which are $\mathcal{S}_i$-measurable such that:*

(a) $E_i(\xi_i) = \nu(x)$ *and* $E_i(\eta_i) = \nu(y), i = 1, 2$*;*
(b) $c(x, y) = \mathrm{cov}(\xi_1, \eta_1), c(y, x) = \mathrm{cov}(\eta_2, \xi_2)$*;*
(c) $(c(x, y))^2 \le c(x, x)c(y, y)$*.*

Proof: If $x, y \in \mathcal{O}$, then $\sigma(x) = \{x_i\}_{i=1}^{n_1}$, $\sigma(y) = \{y_j\}_{j=1}^{n_2}$. Let us denote

$$\Omega_1 = \{(x_i, y_j);\ i = 1, \ldots, n_1, \quad j = 1, \ldots, n_2\},$$

$$\Omega_2 = \{(y_j, x_i);\ j = 1, \ldots, n_2, \quad i = 1, \ldots, n_1\},$$

$$\mathcal{S}_k = 2^{\Omega_k}, \quad k = 1, 2.$$

Then $(\Omega_k, \mathcal{S}_k)$ is the measurable space. Let us denote

$$\xi_1((x_i, y_j)) = x_i, \quad \eta_1((x_i, y_j)) = y_j,$$

$$\xi_2((y_j, x_i)) = x_i, \quad \eta_2((y_j, x_i)) = y_j.$$

If p is an s-map, then from the properties of p follows, that $P_1 = p_{x,y}$ is the probability measure on $(\Omega_1, \mathcal{S}_1)$ and $P_2 = p_{y,x}$ is the probability measure on the measurable space $(\Omega_2, \mathcal{S}_2)$.

(a) It is clear, that

$$P_1(\{\omega \in \Omega_1;\ \xi_1(\omega) = x_i\}) = P_1\{(x_i, y_j);\ j = 1, \ldots, n_2\}.$$

From it follows, that

$$P_1(\{\omega \in \Omega_1;\ \xi_1(\omega) = x_i\}) = P(x(x_i), y(\sigma(y))) = p(x(x_i), 1) = \nu(x(x_i)).$$

Hence

$$P_i(\xi_i = x_i) = \nu(x(x_i))$$

$$P_i(\eta_i = y_j) = \nu(y(y_j))$$

Now, we can see, that

$$E_i(\xi_i) = \sum_k x_k P_i(\xi_i = x_k) = \nu(x).$$

Similarly we get

$$E_i(\eta_i) = \nu(y).$$

(b) From the theory of probability it follows, that

$$\mathrm{var}(\xi_1, \eta_1) = \sum_i \sum_j (x_i - \nu(x))(y_j - \nu(y)) P_1(\xi_1 = x_i,\ \eta_1 = y_j))$$

Let us denote $a_i = x(x_i)$ and $b_j = y(y_j)$. Then

$$\mathrm{cov}(\xi_1, \eta_1) = \sum_{i,j} (x_i - \nu(x))(y_j - \nu(y)) p(a_i, b_j)$$

and

$$\mathrm{cov}(\xi_2, \eta_2) = \sum_{i,j} (x_i - \nu(x))(y_j - \nu(y)) p(b_j, a_i).$$

As

$$\sum_{i,j} p(a_i, b_j) = \sum_{i,j} p(b_j, a_i) = 1,$$

$$\sum_{1} p(a_i, b_j) = \sum_{i} p(b_j, a_i) = \nu(b_j)$$

and

$$\sum_{i} p(a_i, b_j) = \sum_{j} p(b_j, a_i) = \nu(a_i)$$

we have

$$\sum_{i,j} (x_i - \nu(x))(y_j - \nu(y)) p(b_j, a_i) = p(x, y) - \nu(x)\nu(y) = c(x, y).$$

Similarly

$$\text{var}(\xi_2, \eta_2) = p(y, x) - \nu(x)\nu(y) = c(y, x).$$

(c) As $(\text{cov})(\xi_k, \eta_k))^2 \leq \text{cov}(\xi_k, \xi_k)\text{cov}(\eta_k, \eta_k)$ and $\text{cov}(\xi_k, \xi_k) = c(x, x)$, $\text{cov}(\eta_k, \eta_k) = c(y, y)$ we have

$$(c(x, y))^2 \leq c(x, x)c(y, y),$$

$$(c(y, x))^2 \leq c(x, x)c(y, y). \qquad \square$$

Proposition 2.7. *Let L be an OML and let $x, y \in \mathcal{O}$. Then*

(i) $c(x, y) = p(g_x \circ x, g_y \circ y)$, *where g_z is a real function such that $g_z(t) = t - \nu(z)$, for $z \in \mathcal{O}$;*
(ii) $r(x, y) \in [-1, 1]$;
(iii) *if $x \leftrightarrow y$, then $c(x, y) = c(y, x)$ and $r(x, y) = r(y, x)$.*

Proof: Let $x, y \in \mathcal{O}$. Then $\sigma(x) = \{x_i\}_{i=1}^{n_1}$ and $\sigma(y) = \{y_j\}_{j=1}^{n_2}$.

(i) From the definition $g_x \circ x$, $g_y \circ y$ we have

$$p(g_x \circ x, g_y \circ y) = \sum_i \sum_j g_x(x_i) g_y(y_j) p(x(x_i), y(y_j))$$

$$= \sum_i \sum_j (x_i - \nu(x))(y_j - \nu(y)) p(x(x_i), y(y_j))$$

$$= \sum_i \sum_j x_i y_j p(x(x_i), y(y_j))$$

$$= \nu(x)\nu(y)\sum_i \sum_j p(x(x_i), y(y_j))$$

$$= p(x, y) - \nu(x)\nu(y) = c(x, y).$$

(ii) From the previous proposition we know that there exist two probability spaces $(\Omega, \mathcal{F}_k, P_k)$ and random variables $\xi_k, \eta_k, k = 1, 2$ such that $c(x, y) = \mathrm{cov}(\xi_1, \eta_1)$ and $c(x, y) = \mathrm{cov}(\xi_2, \eta_2)$. Moreover $c(x, x) = \mathrm{cov}(\xi_k, \xi_k) = \mathrm{var}(\xi_k)$. Analogically $c(y, y) = \mathrm{cov}(\eta_k, \eta_k) = \mathrm{var}(\eta_k)$. Then

$$r(x, y) = \frac{c(x, y)}{\sqrt{c(x, x)c(y, y)}} = \frac{\mathrm{cov}(\xi_1, \eta_1)}{\sqrt{\mathrm{var}(\xi_1)\mathrm{var}(\eta_1)}} = \rho(\xi_1, \eta_1) = \rho_1$$

and

$$r(y, x) = \frac{c(y, x)}{\sqrt{c(x, x)c(y, y)}} = \frac{\mathrm{cov}(\xi_2, \eta_2)}{\sqrt{\mathrm{var}(\xi_2)\mathrm{var}(\eta_2)}} = \rho(\xi_2, \eta_2) = \rho_2.$$

because ρ_k is correlation coefficient on the probability space $(\Omega_k, \mathcal{F}_k, P_k)$, then $\rho_k \in [-1, 1]$ for $k = 1, 2$. From this follows that $r(x, y), r(y, x) \in [-1, 1]$. Also this fact follows immediately from the Proposition 2.4 (c).

(iii) Let $x \leftrightarrow y$. Then for each $x_i \in \sigma(x)$ and each for $y_j \in \sigma(y)$ we have

$$p(x(x_i), y(y_j)) = p(y(y_j), x(x_i))$$

and so

$$p(x, y) = \sum_i \sum_j x_i y_j p(x(x_i), y(y_j))$$

$$= \sum_i \sum_j x_i y_j p(y(y_j), x(x_i))$$

$$= p(y, x).$$

From this follows that

$$c(x, y) = c(y, x). \qquad \square$$

In the previous proof we could see, than an un-symmetry of a covariance is dependent only on an un-symmetric s-map. Let $(\Omega, \mathcal{S}, P)$ be a classical probability space and ξ, η be some random variables on it. From the classical theory of probability we know, that the set of all random variables is a linear space, the covariance $\mathrm{cov}(\xi, \eta)$ is the inner product and the standard deviation $\sqrt{\mathrm{cov}(\eta, \eta)}$ is the norm. From it follows that the correlation coefficient

$$\rho(\xi, \eta) = \frac{\mathrm{cov}(\xi, \eta)}{\sqrt{\mathrm{cov}(\eta, \eta)\mathrm{cov}(\xi, \xi)}} = \cos(\beta_P),$$

where β_P is the angle between the random variables ξ and η in this geometry. For the example Bell's inequality is valuated only for an un-symmetric s-map also on a Hilbert space. Each Hilbert space is an OML. From Proposition 2.5 it follows, that if f_p is Bayesian, then Bell's inequality is not valuated also for noncompatible observables.

Example 2.1. Let $L = \{a, a^\perp, b, b^\perp, 0, 1\}$. Let $c \vee d = 1$ if $c \neq d$ and $c, d \in L - \{0\}$. Let $c \wedge d = 0$ if $c \neq d$ and $c, d \in L - \{1\}$. $Let (d^\perp)^\perp = d$ for $d \in L$ and $1^\perp = 0$. It is clear that L is an OML and $B_d = \{d, d^\perp, 0, 1\}$, $d \in \{a, b\}$ is a Booelan algebra. Let $f(s, t)$ is defined by the following way:

s/t	a	$a^\perp$	b	$b^\perp$	1
a	1	0	0.4	0.4	0.4
$a^\perp$	0	1	0.6	0.6	0.6
b	0.2	11/30	1	0	0.3
$b^\perp$	0.8	19/30	0	1	0.7

From f this we can compute $p_f(s, t)$. Then we get:

s/t	a	$a^\perp$	b	$b^\perp$
a	0.4	0	0.12	0.28
$a^\perp$	0	0.6	0.18	0.42
b	0.08	0.22	0.3	0
$b^\perp$	0.32	0.38	0	0.7

We can see that $p_f(a, b) = p_f(a, a)p_f(b, b)$, but $p_f(b, a) \neq p_f(b, b)p_f(a, a)$. In the following we will write $p_f = p$. Let x, y be observables on L such that $R(x) = \{a, a^\perp, 0, 1\} = B_a$, and $R(y) = \{b, b^\perp, 0, 1\} = B_b$. It is easy to see, that x is not compatible with y. Let, for example,

$$x(-1) = a \qquad x(1) = a^\perp$$

$$y(0) = b \qquad y(5) = b^\perp.$$

In the following tables we have the joint distributions $p_{x,y}$ and $p_{y,x}$.

$p_{x,y}$	0	5
−1	0.12	0.28
1	0.18	0.42

$p_{y,x}$	−1	1
0	0.08	0.22
5	0.32	0.38

Now we can compute the following characteristics:

$$\begin{aligned}
\nu(x) &= -1 \times 0.4 + 0.2 \times 0.6 = 0.2,\\
\nu(y) &= 0 \times 0.3 + 5 \times 0.7 = 3.5,\\
p(x, y) &= -5 \times 0.28 + 5 \times 0.42 = 0.7,\\
p(y, x) &= -5 \times 0.32 + 5 \times 0.38 = 0.3,\\
c(x, y) &= p(x, y) - \nu(x)\nu(y) = 0.7 - 0.2 \times 3.5 = 0,\\
c(y, x) &= p(y, x) - \nu(x)\nu(y) = 0.3 - 0.2 \times 3.5 = -0.4,\\
c(x, x) &= 0.96 \qquad c(y, y) = 5.25,\\
r(x, y) &= 0 \qquad r(y, x) = 0.178.
\end{aligned}$$

In the end we can rewrite these results in to "the covariance matrix":

$$\begin{pmatrix} c(x, x) & c(x, y) \\ c(y, x) & c(y, y) \end{pmatrix} = \begin{pmatrix} 0.96 & 0 \\ 0.178 & 5.25 \end{pmatrix}$$

We can see that "the covariance matrix" need not be in symmetry. In the classical theory of probability, where we suppose that all random variables are compatible, it has to be in symmetry.

Example 2.2. Let L be the same OML as in the Example 2.1. Let $p(s, t)$ is defined by the following way:

s/t	a	$a^\perp$	b	$b^\perp$
a	0.4	0	0.08	0.38
$a^\perp$	0	0.6	0.22	0.32
b	0.08	0.22	0.3	0
$b^\perp$	0.32	0.38	0	0.7

Let x, y be observables on L such that $R(x) = \{a, a^\perp, 0, 1\}$, and $R(y) = \{b, b^\perp, 0, 1\}$. It is easy to see, that x is not compatible with y. Let, for example,

$$x(-1) = a \qquad x(1) = a^\perp$$

$$y(0) = b \qquad y(5) = b^\perp .$$

In this case $p_{x,y} = p_{y,x}$.

$p_{x,y}$	0	5
-1	0.08	0.38
1	0.22	0.32

Now we get

$$\nu(x) = 0.2, \qquad \nu(y) = 3.5,$$

$$p(x, y) = p(y, x) = -0.3, \qquad c(x, y) = c(y, x) = -0.4,$$

$$c(x, x) = 0.96, \qquad c(y, y) = 5.25,$$

$$r(x, y) = r(y, x) = 0.178.$$

In the end we can write these results to the covariance matrix:

$$\begin{pmatrix} c(x,x) & c(x,y) \\ c(y,x) & c(y,y) \end{pmatrix} = \begin{pmatrix} 0.96 & 0.178 \\ 0.178 & 5.25 \end{pmatrix}$$

We can see that the covariance matrix is symmetry as in the classical theory of probability, but x,y are not compatible.

In the end we can say, that we cannot prove that two observables are compatible by using statistics, but we can only prove that they are not compatible.

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